Groupoid actions as quantale modules*

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Abstract

For an arbitrary localic étale groupoid G we provide simple descriptions, in terms of modules over the quantale $\mathcal{O}(G)$ of the groupoid, of the continuous actions of G, including actions on open maps and sheaves. The category of G-actions is isomorphic to a corresponding category of $\mathcal{O}(G)$-modules, and as a corollary we obtain a new quantale based representation of étendues.

Keywords: localic étale groupoids, groupoid actions, groupoid sheaves, inverse quantales, quantale modules, étendues.

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1 Introduction

Every étale groupoid G, either localic or topological, has an associated unital involutive quantale $\mathcal{O}(G)$ [2]. The class of quantales obtained in this way has been characterized in [2] as consisting of the so-called inverse quantal frames. This provides us with a ring-like description of étale groupoids and it is natural to examine various groupoid related constructions in this light. In this paper we look at continuous actions of étale groupoids from this point of view, and show how they can be identified with a suitable class of quantale modules, in particular obtaining characterizations of groupoid actions on open maps and sheaves.

The module theoretic characterization of groupoid actions obtained is surprisingly simple. If Q is the quantale of an étale groupoid G then a

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left Q-module X corresponds to a G-action if and only if it is a locale and it satisfies the condition $bx = b1 \wedge x$ for all $b \in B$ and $x \in X$, where $B \subset Q$ is the locale of elements below the multiplicative unit e of Q, which is isomorphic to the locale of objects G_0 of the groupoid. This generalizes in the simplest possible way the characterization of locale maps $p : X \to B$ as B-modules: if a left Q-module X defines a map into G_0 at all, then it defines a G-action.

The characterization of open groupoid bundles and groupoid sheaves is then a straightforward consequence of the characterizations of [3], in terms of B-modules, of open maps and local homeomorphisms $p : X \to B$. As regards sheaves, there are in fact two different characterizations in [3]. In the present paper we shall discuss the more straightforward one. The other is based on a notion of quantale module equipped with a quantale-valued "inner product" inspired by the theory of C*-modules. Its application to groupoid sheaves is also rewarding because it leads to very simple axioms and to a theory of sheaves that has interesting properties in the context of quantales that are more general than groupoid quantales. But it requires a lengthier presentation and will appear in a separate paper [4].

The characterizations obtained lead to several isomorphisms of categories, including two module-theoretic descriptions of the classifying topos BG of an étale groupoid G, and in particular provide us with a new representation theorem for étendues, due to [1, Theorem VIII.3.3]: see Theorem 4.11 below.

The rest of the paper is organized into three more sections. In section 2 we discuss a few preliminary results concerning general continuous actions of open groupoids. In section 3 we obtain the main results concerning actions of étale groupoids, and in section 4 we discuss actions on open maps and sheaves.

We shall assume from the reader background knowledge of locales, localic groupoids, inverse semigroups, quantales and their modules, mostly as described in [2, Section 2].

2 Preliminaries on groupoid actions

From now on G is a localic groupoid, i.e., an internal groupoid in the category of locales **Loc**,

$$\begin{tikzcd} G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "r"] \arrow[r, shift right=2, "d"'] & G_0 \arrow[l, "u" description] \end{tikzcd}$$

where as usual G_2 is the pullback of the domain and range maps:

$$\begin{tikzcd} G_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & G_1 \arrow[d, "r"] \\ G_1 \arrow[r, "d"'] & G_0 \end{tikzcd}$$

Groupoid quantales. Let us recall a few basic aspects of the correspondence between groupoids and quantales from [2]. Since G is a groupoid rather than just a category, G_2 is also the pullback of d along itself:

$$\begin{tikzcd} G_2 \arrow[r, "\pi_1"] \arrow[d, "m"'] & G_1 \arrow[d, "d"] \\ G_1 \arrow[r, "d"'] & G_0 \end{tikzcd}$$

Hence, if G is an *open* groupoid (i.e., if d is an open map) then m is an open map and thus there is a sup-lattice homomorphism defined as the following composition (in the category **SL** of sup-lattices and their homomorphisms):

$$G_1 \otimes G_1 \twoheadrightarrow G_2 \xrightarrow{m_!} G_1$$

This defines an associative multiplication on G_1 which together with the isomorphism

$$G_1 \xrightarrow{i_!} G_1$$

makes G_1 an involutive quantale. We denote this quantale by $\mathcal{O}(G)$ — it is the "opens of G".

Actions of open groupoids. The assignment from groupoids to quantales has a straightforward one-sided generalization showing that actions of open groupoids define quantale modules. In order to describe this let us adopt a few more or less standard definitions and terminology. By a *locale over* G_0, or simply a G_0*-locale*, will be meant a locale X together with a map $p : X \to G_0$ called the *projection* into G_0. The *category of* G_0*-locales* is the slice category $\mathbf{Loc}/G_0$. A *(left) action* of G on the G_0-locale (X, p) is a map of locales $\mathfrak{a} : G_1 \times_0 X \to X$ such that the following diagrams commute, where $G_1 \times_0 X$, $G_2 \times_0 X$ and $G_1 \times_0 (G_1 \times_0 X)$ are pullbacks in **Loc** respectively of

r and p, $r \circ \pi_2$ and p, and r and $d \circ \pi_1$:

(2.1)
$$\begin{tikzcd} G_1 \times_0 X \arrow[r, "\pi_1"] \arrow[d, "\mathfrak{a}"'] & G_1 \arrow[d, "d"] \\ X \arrow[r, "p"'] & G_0 \end{tikzcd}$$

(2.2)
$$\begin{tikzcd} G_1 \times_0 (G_1 \times_0 X) \arrow[rr, "1\times\mathfrak{a}"] \arrow[d, "\cong"'] & & G_1 \times_0 X \arrow[dd, "\mathfrak{a}"] \\ G_2 \times_0 X \arrow[d, "m\times 1"'] & & \\ G_1 \times_0 X \arrow[rr, "\mathfrak{a}"'] & & X \end{tikzcd}$$
(Associativity)

(2.3)
$$\begin{tikzcd} & G_1 \times_0 X \arrow[dr, "\mathfrak{a}"] & \\ X \arrow[ur, "\langle u\circ p, 1\rangle"] \arrow[rr, equal] & & X \end{tikzcd}$$
(Unitarity)

The G_0-locale (X, p) together with the action $\mathfrak{a}$ will be referred to as a *(left) G-locale* and we shall denote it by $(X, p, \mathfrak{a})$, or simply by X when no confusion will arise.

The following simple fact will be useful a few times later on:

Lemma 2.4 *Let $p : X \to G_0$ be a map of locales and let $G_1 \times_0 X$ be the pullback of r and p. Then the projection $\pi_1 : G_1 \times_0 X \to G_1$ coincides with the map $m \circ (1 \times (u \circ p))$. In particular, (2.1) is equivalent to the equation $p \circ \mathfrak{a} = d \circ m \circ (1 \times (u \circ p))$.*

Proof. This follows from the commutativity of the following diagram, whose left triangle is obviously commutative and whose right triangle is commutative due to one of the unit laws of G:

$$\begin{tikzcd} G_1 \times_0 X \arrow[r, "1\times p"] \arrow[dr, "\pi_1"'] & G_1 \times_0 G_0 \arrow[r, "1\times u"] \arrow[d, "\cong"', "\pi_1"] & G_1 \times_0 G_1 \arrow[dl, "m"] \\ & G_1 & \end{tikzcd}$$
∎

It is easy to show that the diagram (2.1) is a pullback (briefly, because the action can be reversed due to the inversion operation i of the groupoid), and thus if G is an open groupoid the action map $\mathfrak{a}$ is necessarily open. Hence, in this case, taking into account that $G_1 \times_0 X$ is, in **Frm**, a quotient $G_1 \otimes_0 X$

of the tensor product $G_1 \otimes X$, we obtain a sup-lattice homomorphism by composing with the direct image of the action:

$$G_1 \otimes X \twoheadrightarrow G_1 \otimes_0 X \xrightarrow{\mathfrak{a}_!} X$$

Showing that this defines an action of $\mathcal{O}(G)$ on X (a left quantale module) is straightforward and entirely analogous to the proof of associativity of the quantale multiplication of $\mathcal{O}(G)$ (see [2]).

Definition 2.5 Let G be an open groupoid. We shall denote by $\mathcal{O}(X)$ the left $\mathcal{O}(G)$-module which is obtained from a G-locale X.

Equivariant maps. Let X and Y be G-locales with actions $\mathfrak{a}$ and $\mathfrak{b}$, respectively. An *equivariant map* from X to Y is a map $f : X \to Y$ in $\mathbf{Loc}/G_0$ that commutes with the actions; that is, such that the following diagram commutes:

$$\begin{tikzcd}
G_1 \times_0 X \arrow[r, "1\times f"] \arrow[d, "\mathfrak{a}"'] & G_1 \times_0 Y \arrow[d, "\mathfrak{b}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We shall refer to the category of G-locales and equivariant maps between them as G-$\mathbf{Loc}$. It is simple to see that, since G is a groupoid rather than just a category, the above diagram is actually a pullback. Hence, if G is an open groupoid, in which case as we have seen the actions are open maps, the following diagram in $\mathbf{SL}$ also commutes [1, Proposition V.4.1]:

$$\begin{tikzcd}
G_1 \otimes_0 X \arrow[d, "\mathfrak{a}_!"'] & G_1 \otimes_0 Y \arrow[l, "1\otimes f^*"'] \arrow[d, "\mathfrak{b}_!"] \\
X & Y \arrow[l, "f^*"]
\end{tikzcd}$$

This implies that the locale homomorphism f^* commutes with the actions of $\mathcal{O}(G)$ on $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, and thus it is a homomorphism of $\mathcal{O}(G)$-modules. Hence, we obtain:

Lemma 2.6 *The assignments $X \mapsto \mathcal{O}(X)$ and $f \mapsto f^*$ define a faithful functor $\mathcal{O} : G\text{-}\mathbf{Loc} \to \mathcal{O}(G)\text{-}\mathbf{Mod}^{op}$.*

Comparing this with [2, Example 5.14] we see that the assignment from groupoid actions to modules has better functorial properties than the assignment from groupoids to quantales.

This functor is not full, of course, but we make the following observation:

Lemma 2.7 *Let G be an open groupoid and let $f : X \to Y$ be a map of locales such that f^* is a homomorphism of $\mathcal{O}(G)$-modules. Denoting the actions of X and Y by $\mathfrak{a}$ and $\mathfrak{b}$, respectively, we have $f \circ \mathfrak{a} \geq \mathfrak{b} \circ (1 \times f)$.*

Proof. Let us prove the inverse image version of the inequality, that is

$$\mathfrak{a}^* \circ f^* \geq (1 \otimes f^*) \circ b^* \ ,$$

using the equality $f^* \circ \mathfrak{b}_! = \mathfrak{a}_! \circ (1 \otimes f^*)$ that corresponds to the Q-equivariance of f^*:

$$\mathfrak{a}^* \circ f^* \geq \mathfrak{a}^* \circ f^* \circ \mathfrak{b}_! \circ \mathfrak{b}^* = \mathfrak{a}^* \circ \mathfrak{a}_! \circ (1 \otimes f^*) \circ \mathfrak{b}^* \geq (1 \otimes f^*) \circ \mathfrak{b}^* = (1 \times f)^* \circ \mathfrak{b}^* \ . \quad \blacksquare$$

3 Actions of étale groupoids

The groupoid G

$$\begin{tikzcd} G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "r"] \arrow[r, shift right=2, "d"'] & G_0 \arrow[l, "u"'] \end{tikzcd}$$

is said to be *étale* if d (equivalently r or m) is a local homeomorphism. This is equivalent [2] to G being an open groupoid whose units map u is also open (hence, succinctly, a groupoid is étale if and only if all of its structure maps are open). Moreover, if G is étale the involutive quantale $\mathcal{O}(G)$ is unital with unit $e = u_!(1_{G_0})$, and the downsegment

$$\downarrow(e) = \{b \in \mathcal{O}(G) \mid b \leq e\}$$

is both a unital involutive subquantale of $\mathcal{O}(G)$ (with trivial involution $b^* = b$) and it is a locale isomorphic to G_0 (it is the image of $u_!$) whose multiplication ab coincides with binary meet $a \wedge b$ [2].

Q-locales. From now on G is an arbitrary but fixed étale groupoid. We shall denote the quantale $\mathcal{O}(G)$ by Q and the locale $\downarrow(e)$ by B (the "base locale"). If X is a G-locale with projection $p : X \to G_0$ then X is a G_0-module by change of "ring" along the inverse image homomorphism $p^* : G_0 \to X$. The same action of G_0 on X can be obtained through the isomorphism $G_0 \cong B$ by restricting the action of Q:

Lemma 3.1 *Let X be a G-locale with projection $p : X \to G_0$. For all $b \in G_0$ and $x \in X$ we have $u_!(b)x = p^*(b) \wedge x$. In particular, $\mathcal{O}(X)$ is a unital Q-module and the action uniquely defines p by the equation $p^*(b) = u_!(b)1$.*

Proof. Axiom (2.3) of G-locales is $\mathfrak{a} \circ \langle u \circ p, 1 \rangle = 1$, which we can rewrite as $\mathfrak{a} \circ (u \times 1) \circ \langle p, 1 \rangle = 1$, where the pairing $\langle p, 1 \rangle : X \to G_0 \times_0 X$ is an isomorphism and thus $\mathfrak{a} \circ (u \times 1) = \langle p, 1 \rangle^{-1}$. Hence, we have

$$\mathfrak{a}_! \circ (u_! \otimes 1) = [p^*, 1]$$

and the required equation follows:

$$\begin{aligned} u_!(b)x &= \mathfrak{a}_!(u_!(b) \otimes x) = (\mathfrak{a}_! \circ (u_! \otimes 1))(b \otimes x) \\ &= [p^*, 1](b \otimes x) = p^*(b) \wedge x \; . \quad \blacksquare \end{aligned}$$

Hence, the faithful functor $\mathcal{O} : G\text{-}\mathbf{Loc} \to Q\text{-}\mathbf{Mod}^{\mathrm{op}}$ of 2.6 restricts to a functor to the following category Q-**Loc**:

Definition 3.2 By a Q-*locale* will be meant a locale X which is also a unital left Q-module whose action satisfies the condition $bx = b1 \wedge x$ for all $b \in B$ and $x \in X$. The *category of* Q-*locales*, Q-**Loc**, is that whose objects are the Q-locales and whose morphisms $f : X \to Y$ are the maps of locales such that f^* is a homomorphism of Q-modules.

Example 3.3 Q itself is a Q-locale, since (G, d, m) is a G-locale: the equality $ba = b1 \wedge a$ holds for all $b \in B$ and $a \in Q$, and, due to the involution, $ab = 1b \wedge a$ also holds (corresponding to the right G-locale structure of G with projection r). More generally, these are general properties of the stably supported quantales of [2].

Example 3.4 If X is a B-locale then $Q \otimes_B X$ is a locale whose natural left Q-action makes it a Q-locale:

$$b(a \otimes x) = ba \otimes x = (b1 \wedge a) \otimes x = b(1 \otimes 1) \wedge (a \otimes x) \; .$$

If X corresponds to a G_0-locale $p : X \to G_0$ then the Q-locale $Q \otimes_B X$ corresponds to a G-locale $G_1 \times_0 X$ whose projection $d \circ \pi_1$ (where π_1 is the pullback of p along r) is an open map (resp. a local homeomorphism) if p is.

Example 3.5 If Q coincides with the locale B (i.e., the groupoid G is just the locale $G_1 = G_0$ with identity structure maps) the categories B-**Loc** and **Loc**$/B$ are easily seen to be isomorphic [3]: the isomorphism sends each map $p : X \to B$ to the module $\mathcal{O}(X)$ whose action is defined by $bx = p^*(b) \wedge x$ and, conversely, knowing the action one defines p by the formula $p^*(b) = b1_X$; a map of locales $f : X \to Y$ is in **Loc**$/B$ if and only if it is in B-**Loc**.

Multiplicativity. Now let us generalize the latter example to more general quantales. In particular, as we shall see, every Q-locale arises from a G-locale. We begin by observing that any unital left Q-module X (not necessarily a Q-locale, or even a locale) is also a unital left B-module due to the inclusion $B \to Q$. Hence, we can form the tensor product $Q \otimes_B X$. The associativity of the action $Q \otimes X \to X$ implies that it factors through the quotient $Q \otimes X \to Q \otimes_B X$ and a sup-lattice homomorphism $\alpha : Q \otimes_B X \to X$, whose right adjoint α_* is given by

$$\alpha_*(x) = \bigvee \{a \otimes y \in Q \otimes_B X \mid \alpha(a \otimes y) \le x\} \tag{3.6}$$

$$= \bigvee \{a \otimes y \in Q \otimes_B X \mid ay \le x\} \ . \tag{3.7}$$

The fact that $Q = \mathcal{O}(G)$ for an étale groupoid G (in other words, Q is an inverse quantal frame [2]) provides us with a more useful formula for α_*. In order to see this we first recall that the local bisections of G form an inverse semigroup and they can be identified [2] with the *partial units* of Q, that is the elements $s \in Q$ such that $\{ss^*, s^*s\} \subset B$, which also satisfy $ss^*s = s$. The set $\mathcal{I}(Q)$ of partial units of Q is also a basis in the locale sense and it is downwards closed. In particular we have $\bigvee \mathcal{I}(Q) = 1$.

Lemma 3.8 *Let X be a unital left Q-module with action $\alpha : Q \otimes_B X \to X$. The right adjoint α_* is given by, for all $x \in X$,*

$$\alpha_*(x) = \bigvee_{s \in \mathcal{I}(Q)} s \otimes s^* x \ . \tag{3.9}$$

It follows that α_* preserves arbitrary joins (besides arbitrary meets).

Proof. Since $\mathcal{I}(Q)$ is join-dense in Q and joins distribute over tensors we can equivalently replace a in (3.7) by $s \in \mathcal{I}(Q)$ and thus obtain

$$\begin{aligned} \alpha_*(x) &= \bigvee_{sy \le x} s \otimes y \ \le \bigvee_{s^*sy \le s^*x} s \otimes y \ = \bigvee_{s^*sy \le s^*x} ss^*s \otimes y \\ &= \bigvee_{s^*sy \le s^*x} s \otimes s^*sy \quad \text{(because } s^*s \in B\text{)} \\ &\le \bigvee_{s \in \mathcal{I}(Q)} s \otimes s^*x \ \le \ \alpha_*(x) \ , \end{aligned}$$

where the last inequality is a consequence of the fact that for each $s \in \mathcal{I}(Q)$ we have $ss^*x \le x$ and thus $s \otimes s^*x \le \alpha_*(x)$. Hence, all the above

inequalities are in fact equalities. The fact that α_* preserves joins is an immediate consequence, for if $Y \subset X$ then

$$\alpha_*\left(\bigvee Y\right) = \bigvee_{s\in\mathcal{I}(Q)} s\otimes s^*\bigvee Y = \bigvee_{x\in Y}\bigvee_{s\in\mathcal{I}(Q)} s\otimes s^* x = \bigvee \alpha_*(Y)\ . \blacksquare$$

Remark 3.10 This result holds under more general assumptions, namely it suffices that Q be a unital involutive quantale containing a join-dense sub-involutive-semigroup $S \subset Q$ such that $ss^* \le e$ and $s \le ss^*s$ (hence, $s = ss^*s$) for all $s \in S$ (notice that $B = \downarrow(e)$ is always a unital involutive subquantale of Q and the same remarks about the tensor product $Q \otimes_B X$ apply). In this more general situation we obtain

$$\alpha_*(x) = \bigvee_{s\in S} s\otimes s^* x\ .$$

Examples of such quantales are the inverse quantales of [2] — the set $\mathcal{I}(Q)$ of partial units of an inverse quantale Q is a join-dense complete inverse monoid whose locale of idempotents coincides with B. Such a quantale is of the form $\mathcal{O}(G)$ for an étale groupoid G if and only if it is also a locale (an inverse quantal frame) [2]. As a corollary of this we conclude that the multiplication $\mu : Q \otimes_B Q \to Q$ of an inverse quantale Q necessarily has a join preserving right adjoint given by

$$\mu_*(a) = \bigvee_{s\in\mathcal{I}(Q)} s\otimes s^* a \tag{3.11}$$

In particular, we obtain in this way a new proof of the fact that every inverse quantal frame is multiplicative [2].

Equivalence between G-locales and Q-locales. Now we shall see that the categories of G-locales and of Q-locales amount to the same thing.

Lemma 3.12 *The assignment $X \mapsto \mathcal{O}(X)$ from G-locales to Q-locales is a (strict) bijection.*

Proof. Let X be a Q-locale. The inclusion $B \subset Q$ makes X a B-locale and thus we have a map $p : X \to G_0$ defined by $p^*(b) = u_!(b)1$ (cf. 3.5). Since the pullback $G_1 \times_0 X$ of r and p is, in the category of frames, the quotient of the frame coproduct $G_1 \otimes X$ generated by the equalities

$$\pi_1^*(r^*(b)) = \pi_2^*(p^*(b))\ , \tag{3.13}$$

the Q-locale conditions $p^*(b) \wedge x = u_!(b)x$ and $a \wedge r^*(b) = au_!(b)$ (cf. 3.3) show, if we stabilize (3.13) under finite meets, that $G_1 \times_0 X$ coincides with the sup-lattice quotient generated by the equalities $au_!(b) \otimes x = a \otimes u_!(b)x$, in other words it is the tensor product of B-modules $Q \otimes_B X$. Since the right adjoint α_* of the module action

$$\alpha : Q \otimes_B X \to X$$

preserves joins (see 3.8) we define a groupoid action $\mathfrak{a} : G_1 \times_0 X \to X$ by $\mathfrak{a}^* = \alpha_*$ and in order to see that we have obtained a G-locale all we need is to verify that the three axioms (2.1)–(2.3) are satisfied. Of course, once this is done our proof will be finished because it is clear that the construction of the G-locale structure from the Q-locale thus obtained is the inverse of the assignment $Y \mapsto \mathcal{O}(Y)$.

Axiom (2.2) (the associativity of $\mathfrak{a}$) follows in a straightforward manner from the associativity of α because $\alpha = \mathfrak{a}_!$. (This is completely analogous to the way in which the associativity of the multiplication of an open groupoid follows from the associativity of the multiplication of its quantale.)

Proving the two other axioms is less easy because p is not necessarily an open map and thus we do not have straightforward direct image versions of the axioms we want to prove. Let us start with axiom (2.1). By 2.4, this is equivalent to the equation $p \circ \mathfrak{a} = d \circ m \circ (1 \times (u \circ p))$, which we can verify directly in terms of inverse images using the formulas (3.9) and (3.11) for $\mathfrak{a}^*$ and m^*: on one hand we have

$$\mathfrak{a}^*(p^*(b)) = \bigvee_{s \in \mathcal{I}(Q)} s \otimes s^* u_!(b) 1_X$$

and, on the other,

$$m^*(d^*(b)) = \bigvee_{s \in \mathcal{I}(Q)} s \otimes s^* u_!(b) 1_Q \ .$$

The inverse image of $1 \times (u \circ p)$ is given by

$$(1 \otimes (p^* \circ u^*))(a \otimes c) = a \otimes ((c \wedge e) 1_X)$$

and, combining these formulas, we obtain

$$1 \otimes (p^* \circ u^*)(m^*(d^*(b))) = \bigvee_{s \in \mathcal{I}(Q)} s \otimes (s^* u_!(b) 1_Q \wedge e) 1_X = \mathfrak{a}^*(p^*(b)) \ ,$$

where the last step follows from the following three facts: (i) $s^* u_!(b)$ belongs to $\mathcal{I}(Q)$; (ii) for all $t \in \mathcal{I}(Q)$ we have $t 1_Q \wedge e = tt^*$ [2]; (iii) for all $t \in \mathcal{I}(Q)$ we have $tt^* 1_X \leq t 1_X = tt^* t 1_X \leq tt^* 1_X$, and thus $(s^* u_!(b) 1_Q \wedge e) 1_X = s^* u_!(b) 1_X$.

Now let us verify axiom (2.3). The inverse image of $\mathfrak{a} \circ \langle u \circ p, 1\rangle$ is given by

$$[p^* \circ u^*, 1](\mathfrak{a}^*(x)) = \bigvee_{s \in \mathcal{I}(Q)} p^*(u^*(s)) \wedge s^*x = \bigvee_{s \in \mathcal{I}(Q)} (s \wedge e)1_X \wedge s^*x \;.$$

Since X is a Q-locale we have $(s \wedge e)1_X \wedge s^*x = (s \wedge e)s^*x$ and, since s is in the inverse monoid $\mathcal{I}(Q)$, we also have $(s \wedge e)s^* = s \wedge e$. Hence,

$$\bigvee_{s \in \mathcal{I}(Q)} (s \wedge e)1_X \wedge s^*x = \bigvee_{s \in \mathcal{I}(Q)} (s \wedge e)x = \left(\bigvee \mathcal{I}(Q) \wedge e\right) x = ex = x$$

and we conclude that $\mathfrak{a} \circ \langle u \circ p, 1\rangle = 1$ as required. ∎

Theorem 3.14 *The categories G-**Loc** and Q-**Loc** are isomorphic.*

Proof. All we need to do is show that the functor $\mathcal{O} : G\text{-}\mathbf{Loc} \to Q\text{-}\mathbf{Loc}$ is full. Let X and Y be G-locales, let $f : X \to Y$ be a map of locales such that f^* is a homomorphism of Q-modules, and let the actions of G on X and Y be $\mathfrak{a}$ and $\mathfrak{b}$, respectively. By 2.7, in order to prove that the functor is full we only have to prove, for all $y \in Y$, the inequality

$$\mathfrak{a}^*(f^*(y)) \le (1 \otimes f^*)(\mathfrak{b}^*(y)) \;. \tag{3.15}$$

From 3.8 and the fact that f^* is Q-equivariant we have

$$\mathfrak{a}^*(f^*(y)) = \bigvee_{s \in \mathcal{I}(Q)} s \otimes s^*(f^*(y)) = \bigvee_{s \in \mathcal{I}(Q)} s \otimes f^*(s^*y) \;.$$

The expression $s \otimes f^*(s^*y)$ on the right equals $(1 \otimes f^*)(s \otimes (s^*y))$, and we have $s \otimes (s^*y) \le \mathfrak{b}^*(y)$ because $\mathfrak{b}_!(s \otimes (s^*y)) = ss^*y \le y$. This proves the inequality (3.15). ∎

4 Open maps and sheaves

Now let us examine the categories of Q-locales that correspond to G-locales whose projections are open maps or local homeomorphisms.

Actions on open maps. By an *open* G-locale will be meant a G-locale whose projection is an open map. Similarly, the corresponding Q-locales will be called *open*. Their description is very simple and does not even require the Q-locale condition:

Lemma 4.1 *Let X be a Q-module which is also a locale. Then X is an open Q-locale if and only if there exists a (necessarily unique) homomorphism of B-modules*

$$\varsigma : X \to B$$

such that $\varsigma(x)x = x$ for all $x \in X$.

Proof. This follows immediately from the description of open maps of locales $p : X \to B$ in terms of B-modules, as in [3]: if p is open, the homomorphism ς equals $u_! \circ p_!$. ∎

Example 4.2 If X is an open B-locale then $Q \otimes_B X$ is an open Q-locale (cf. 3.4). Its support is defined by $\varsigma(a \otimes x) = \varsigma(a\varsigma(x))$.

If X is an open Q-locale and $x \in X$, we shall refer to $\varsigma(x)$ as the *support* of x, and ς itself will be said to be the *support* of X. This terminology is analogous to that of [2] for groupoid quantales: the direct image homomorphisms $d_!$ and $u_!$ define a homomorphism of left B-modules $\varsigma = u_! \circ d_! : Q \to B$ that satisfies $\varsigma(a)a = a$ and $\varsigma(a) \le aa^*$ for all $a \in Q$, and also derived properties such as $B = \varsigma(Q)$, $\varsigma(a) = \varsigma(a)^* = \varsigma(a)\varsigma(a)$, $\varsigma(ab) = \varsigma(a\varsigma(b))$ and $\varsigma(ab) \le \varsigma(a)$ for all $a, b \in Q$.[1] We shall use the same notation for the supports of Q and X but the distinction will always be clear. The following are useful properties of open Q-locales:

Theorem 4.3 *Let X be an open Q-locale.*

1. *$\varsigma(ax) = \varsigma(a\varsigma(x))$ for all $a \in Q$ and $x \in X$.*
2. *$\varsigma(ax) \le \varsigma(a)$ for all $a \in Q$ and $x \in X$.*
3. *$\varsigma(sx) = s\varsigma(x)s^*$ for all $s \in \mathcal{I}(Q)$ and $x \in X$.*

Proof. Denoting by p and $\mathfrak{a}$ the projection and the action of the corresponding G-locale and using the equality $p \circ \mathfrak{a} = d \circ m \circ (1 \times (u \circ p))$ of 2.4 we prove 1:

$$\varsigma(ax) = (u \circ p \circ \mathfrak{a})_!(a \otimes x) = (u \circ d \circ m \circ (1 \times (u \circ p)))_!(a \otimes x) = \varsigma(a\varsigma(x)) \ .$$

Then 2 follows immediately: $\varsigma(ax) = \varsigma(a\varsigma(x)) \le \varsigma(ae) = \varsigma(a)$; and 3 is a consequence of the inequalities $s\varsigma(x)s^* \le ss^* \le e$ and

$$\begin{aligned} \varsigma(sx) &= \varsigma(s\varsigma(x)) \le (s\varsigma(x))(s\varsigma(x))^* = s\varsigma(x)s^* \\ &= \varsigma(s\varsigma(x)s^*) \le \varsigma(s\varsigma(x)) \\ &= \varsigma(sx) \ . \end{aligned}$$ ∎

[1] The fact that $\varsigma(ab) = \varsigma(a\varsigma(b))$ is a derived property is not mentioned in [2], but it follows from 4.3(1) that this equation holds for any homomorphism of left B-modules $\varsigma : Q \to B$ that satisfies $\varsigma(a)a = a$.

G**-sheaves.** A G-*sheaf* is a G-locale whose projection is a local homeomorphism. The full subcategory of G-**Loc** whose objects are the G-sheaves (the classifying topos of G) is usually denoted by BG and the isomorphism G-**Loc** $\cong$ Q-**Loc** yields, by restriction, a corresponding category Q-**LH** of *étale Q-locales*. We shall study this along with an isomorphic category Q-**Sh** of "Q-sheaves", whose morphisms are the direct images of the morphisms of Q-**LH**.

If X is a Q-locale then it is also a B-locale. This corresponds to a map $p : X \to B$, which is a local homeomorphism if and only if X is an étale B-locale in the sense of [3]. The subcategory Q-**LH** of Q-**Loc** that corresponds to BG is therefore the full subcategory of Q-**Loc** whose objects, seen as B-modules, are étale B-locales. For the record, we rewrite the definitions of [3], now for Q-locales, and remark that a "local section" s is the same as the image of an actual local section $\bar{s} : U \to X$ of p, where $U \cong \downarrow(\varsigma(s)) \subset B$ is an open sublocale of B:

Definition 4.4 Let X be an open Q-locale. By a *local section* of X is meant an element $s \in X$ such that $x = \varsigma(x)s$ for all $x \leq s$. The set of local sections of X is denoted by $\mathit{\Gamma}_X$ and X is called an *étale Q-locale* if $\bigvee \mathit{\Gamma}_X = 1$. The full subcategory of Q-**Loc** whose objects are the étale Q-locales is denoted by Q-**LH**.

Example 4.5 Q itself is an étale Q-locale and we have $\mathcal{I}(Q) \subset \mathit{\Gamma}_Q$. The partial units $s \in \mathcal{I}(Q)$ are identified with the local *bisections* of G, which are the local sections $\bar{s} : U \to G_1$ of d such that $r \circ \bar{s} : U \to G_0$ is a regular monomorphism in **Loc**.

Example 4.6 If X is an étale B-locale then $Q \otimes_B X$ is an étale Q-locale (cf. 4.2).

An alternative notion of morphism of étale Q-locales, which maps local sections to local sections in the same way that a natural transformation between sheaves does, is the following:

Definition 4.7 Let X and Y be étale Q-locales. A *sheaf homomorphism* $h : X \to Y$ is a homomorphism of left Q-modules that preserves supports and local sections; that is,

$$\varsigma(h(x)) = \varsigma(x) \text{ for all } x \in X \tag{4.8}$$

$$h(\mathit{\Gamma}_X) \subset \mathit{\Gamma}_Y . \tag{4.9}$$

The category of étale Q-locales and sheaf homomorphisms between them is denoted by Q-**Sh**.

Example 4.10 If Q is just the locale B we obtain the category B-**Sh**, which is isomorphic to B-**LH** [3]. The sheaf homomorphisms $h : X \to Y$ are precisely the direct images $f_!$ of the maps $f : X \to Y$ of étale B-locales.

Theorem 4.11 *The categories* BG*,* Q-**LH** *and* Q-**Sh** *are isomorphic.*

Proof. Let X and Y be G-sheaves with actions $\mathfrak{a}$ and $\mathfrak{b}$, respectively. If $f : X \to Y$ is a map of G-sheaves then f is a local homeomorphism and $f_!$ is necessarily a sheaf homomorphism of étale B-locales. From the equivariance condition

$$f \circ \mathfrak{a} = \mathfrak{b} \circ (1 \times f) \tag{4.12}$$

we obtain, passing to direct images, the condition

$$f_! \circ \mathfrak{a}_! = \mathfrak{b}_! \circ (1 \otimes f_!) \tag{4.13}$$

and thus $f_!$ is also a homomorphism of Q-modules. Therefore the assignment $f \mapsto f_!$ defines a faithful functor $F : Q\text{-}\mathbf{LH} \to Q\text{-}\mathbf{Sh}$ which is the identity on objects.

Now let $h : X \to Y$ be an arbitrary sheaf homomorphism of étale Q-locales. This is also a sheaf homomorphism of étale B-locales and thus it is the direct image $f_!$ of a locale map $f : X \to Y$. The Q-equivariance of h is therefore the condition (4.13) and we obtain the inverse image homomorphism version of (4.12) by taking right adjoints. Hence, F is full. ∎

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